Numerical Study of Coulomb and Viscous Damping

Jinyu Hu

Numerical study of Coulomb and viscos

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List of Symbols and Abbreviations

FFT	Fast Fourier transform
p	Time Period
Q	Quality Factor
c	Damping Constant
k	Stiffness Constant
x	Displacement
ω_n	Natural Frequency
ζ	Zeta Damping Coefficient
ω_d	Damped Natural Frequency
ODE	Ordinary Differential Equation

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Author:	Jinyu Hu
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Supervisor (Arcada):	Rene Herrmann
Commissioned by:	

Abstract:

Mechanical vibration damping is broadly classified into viscous and coulomb damping. The objective of the thesis was to study response and characteristic parameters of the vibration damping such as decay factor, resonant frequency, and quality factor. These factors play crucial role in designing and operation of machines. Vibration response and quality factor i.e. 2p times ratio of total energy to damping capacity of viscous, coulomb and combine damping is studied using SCILAB and fast Fourier transform technique is applied to determine the natural frequency of the system. The governing equations are simulated for time response and its Fast Fourier transform FFT is plotted to get the quality factor. Study shows that the quality factor for viscous damped system is lesser than the coulomb damped system. Analysis of the combine damped system shows the dominance of viscous damping in a system as compare to the coulomb damping. This is because viscous damping has greater damping ratio than coulomb damping. The response of combine damped system show the dominance of the viscous damping in damping the system vibration. A sine wave was achieved with the decay curve resembling closely to the overdamped system.

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1 INTRODUCTION

Damping is a phenomenon in which the amplitude of vibration in mechanical systems steadily diminishes. The damping effect occurs by removing the energy from the system. This energy is dissipated either to the surrounding or material of the system itself. A lot of research effort has gone into the investigation of damping and understanding it. Nevertheless, the complexity of the damping phenomenon prevents to present a clear definition of the mechanism by which the vibrational energy is being dissipated [1].

Every system that possesses mass and elasticity is capable of oscillating; therefore, damping is present in every real oscillatory system. Some typical examples include swaying of building due to earthquakes, vibration in automobiles on the road, vibration in the human ear due to sound, etc. The elastic component of the oscillatory system stores energy in the form of potential energy and release kinetic energy in the form of motion. Damping in the system dissipates a small amount of this kinetic energy in each cycle of vibration. It results in resting the body in an equilibrium position. Therefore, the damping phenomenon plays an essential role in the motion of oscillatory systems.

1.1 AIMS AND OBJECTIVE

This thesis concerns the damping of mechanical vibrations. This study investigates combinations of viscous and Coulomb Damping and its influence on the shape of the spectral peak and associated Quality factor (Q) value. The study is based on numerical solutions of the problem.

2 LITERATURE REVIEW

2.1 HISTORY OF VIBRATIONS

A study on vibrations started when people started taking their interest in music, probably drums and whistles. Greek philosopher Pythagoras (582-507 B.C) is the first one who investigated musical vibrations on a scientific basis. He also conducted experiments on vibrating strings using simple apparatus called monochord.

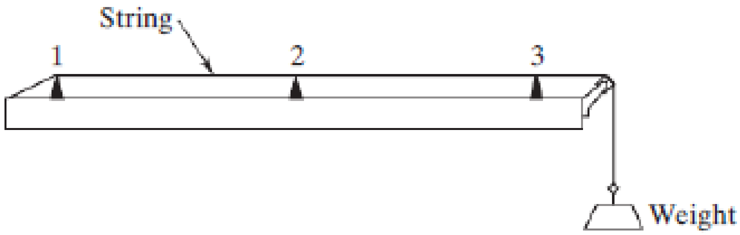

Figure 2.1 Monochord

Zhang Heng served as an astronomer and historian in China in the 2nd century. China faced many earthquakes in those times. In A.D 132, Zhang Heng invented the first seismograph to measure earthquake vibrations [2].

Galileo Galilei is considered the founder of modern experimental science. He studied the behavior and motion of a simple pendulum. He discussed vibrating bodies in his book, which was published in 1638 [5]. French mathematician Marin Mersenne was the first one who published the first correct account on vibration in strings in his book Harmonicorum Liber [6].

Sir Isaac Newton published his book Philosophiae Naturalis Principia Mathematica, in 1686 [7]. He described universal gravitational law as well as laws of motion. Newton's second law of motion is widely used in modern books to derive vibrating bodies equations of motion. English mathematician Brook Taylor gave the theoretical dynamical solution of vibrating string in 1713. Daniel Bernoulli, Jean D’Alembert, and Leonard Euler perfected Taylor’s procedure using partial differential equations.

Joseph Lagrange, in 1759, presented the analytical solution of vibrating string. Charles Coulomb did experimental and theoretical studies on torsional oscillations of metal cylinder suspended through a wire. Lord Baron Rayleigh published his book in 1877, which was based on the theory of sound, in which he gave the method of finding the fundamental frequency of vibration – popularly known as Rayleigh’s method. [6]

2.2 TYPES OF DAMPING

There are different mechanisms of damping by which energy is dissipated so that undesired vibrations can be attenuated. Some types include viscous damping, coulomb damping, hysteresis damping, material damping, etc. [7]

Viscous damping is influenced by energy losses such as those that arise in liquid lubrication among moving components or in a fluid that is pushed into a narrow piston opening, such as in automotive shock absorbers. The viscous damping force is directly proportional to the relative velocity between both two endpoints of the damping system. [2]

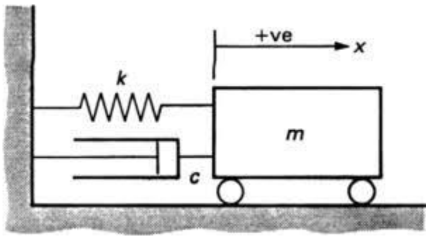

Figure 2.2 Viscous Damping [7]

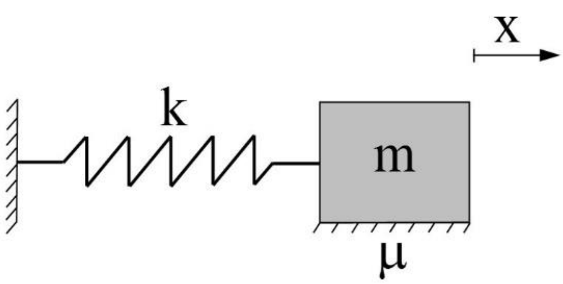

Figure 2.3: Damping [7]

Coulomb damping is a form of continuous mechanical damping where energy is consumed by sliding friction. The friction produced by the relative motion of the two surfaces, which are pressed against each other, is a source of energy dissipation. Damping is the dissipation of energy from a vibrating mechanism where the kinetic energy is transformed into heat by friction. Coulomb damping is a rising damping system that usually occurs in machines [2]. In addition to these external forms of damping, there is a loss of energy within its moving structure itself, which is known as hysteresis damping or, often, structural damping.

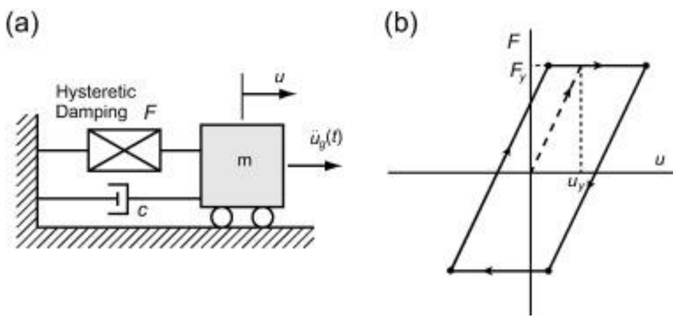

Figure 2.3 Coulomb Damping cycle [7]

In hysteresis damping, a few of the energy consumed in repeated inner deformation and restoring to the original form is dissipated in the form of random crystal lattice vibrations in solids and random kinetic energy of the molecules in the fluid. [8]

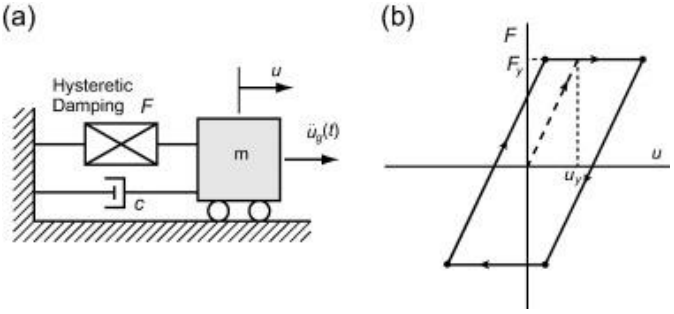

Figure 2.4 Hysteresis Damping System [4]

2.3 QUALITY FACTOR

In physics and engineering, the quality factor or Q factor is a dimensionless parameter that describes how under-damped is an oscillator. It characterizes a resonator's bandwidth relative to its center frequency [9]. Higher Q indicates a lower rate of energy loss relative to the stored energy of the resonator; the oscillations die out more slowly. A pendulum suspended from a high-quality bearing, oscillating in air, has a high Q, while a pendulum immersed in oil has a low one. The resonators with top quality factors do have lower damping effects such that the ring or vibrate for a longer period. [2]

2.4 FAST FOURIER TRANSFORM

Fast Fourier Transform (FFT) is a combination of two words 'fast' and 'Fourier transform.' Fast refer to the computational speed and the simplicity of the algorithm used for performing a specific task. On the other hand, Fourier transform refers to the discrete transformation applied to study the general functions in terms of the trigonometric function. It deconstructs the signal into the individual wave components. The original concept of Fourier Transform as evaluated over time and its area of application has increased, and it developed into a general field known as harmonic analysis. It allows the user to perform analysis of acceleration, velocity, and amplitude of the signal in the frequency domain.

3 METHOD

In this project, a model built in which both damping phenomenon, viscous Damping and Coulomb Damping, coexist. The amount of each damping was parametrized to find its contribution to the overall damping by some precentral amount. It will be stimulated in the frequency domain to establish Q value.

In this numerical analysis, two main types of damping systems, i.e., mass-spring viscous damper system and mass-spring coulomb damper system separately, were investigated. In coulomb damping, only dry coulomb damping was considered. Moreover, any other type of damping, such as structural damping, was also not considered.

3.1 MASS SPRING VISCOUS DAMPER SYSTEM

Viscous damping force is related to the different physical attributes such as volume, object shape, and velocity through the viscous media. The more the velocity of the object, the higher the viscous damping force. [7]

Some of the typical examples of viscous damping are the following:

- The flow of the fluid through a hole
- The flow of the fluid within the bearing of the journal
- The flow of fluid around a piston in a cylinder
- Fluid films between the surfaces

Viscous damping is a type of damping in which damping force is directly proportional to the velocity of the system. It can be expressed as, [7]

$$F = -c\dot{x} \tag{1}$$

Here c is constant of damping, and the negative sign shows that damping force is in the opposite direction of the direction of velocity. We will consider only vicious linear damping, as it can be seen from the above equation.

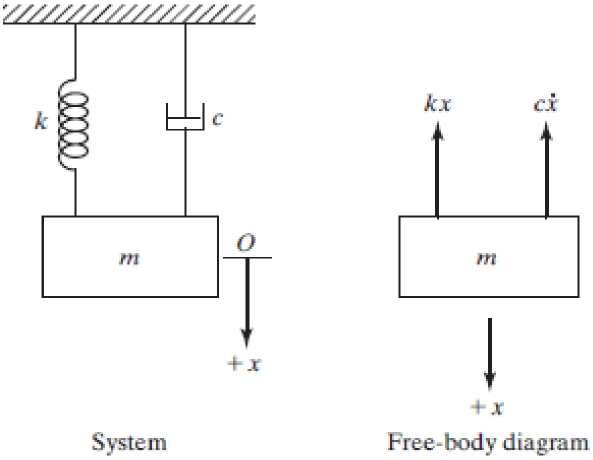

Figure 3.1: Spring Mass SDOF

A single degree of freedom system with a mass, spring, and a viscous damper is considered as shown in the above figure. Take x as the equilibrium position of mass then according to Newton second law of motion the equation of motion: [7]

$$m\ddot{x} = -c\dot{x} - kx \tag{2}$$

It gives us the differential equation that motion of the system. Where m is the mass of the object, c is the damping coefficient, k is the stiffness, and x is the displacement of the body.

$$m\ddot{x} + c\dot{x} + kx = 0$$

Rearranging,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{3}$$

In the above equation, the term ω_n describes the **natural frequency** of the system and has units of rads/sec.

$$\omega_n = \sqrt{\frac{k}{m}}$$

Similarly, the term ζ is called the **damping ratio** of the system. It is dimensionless, indicating the damping level and type of motion of the system.

$$\zeta = \frac{\text{actual damping}}{\text{critical damping}}$$

$$\zeta = \frac{c}{c_c}$$

According to Singiresu S. Rao, assume an exponential solution of the form [7]

$$x(t) = Ce^{st}$$

Where C and s are undetermined constants. The equation of motion of the system can be rewritten as follows:

$$ms^2 + cs + k = 0$$

Solving the above equation and obtaining the roots of the equation as follows:

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{4}$$

These roots give two solutions:

$$x_1(t) = C_1 e^{s_1 t}$$

$$x_2(t) = C_2 e^{s_2 t}$$

Therefore, a general solution for the free damped system by combining the above two solutions. [7]

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{5}$$

The behavior of the solution and nature of roots depend upon the damping ratio. If the damping ratio is zero, then the system is undamped. There are four cases of viscous damping depending upon damping ratio as follows:

3.1.1 Viscous damping cases:

There are four basic cases of viscous damping. To elaborate on the difference between each case, we will take initial perturbation displacement of the system as x_0 and initial perturbation velocity of the system as V_0.

3.1.2 ζ = 0: Undamped

This case is not related to our experiment. An undamped system oscillates about the mean position continuously unless some external force is applied to it. It is a non-realistic type of system in which there is no frictional loss or any other kind of energy dissipation.

3.1.3 ζ < 1: Underdamped

For an undamped system,

$$c^2 < 4mk$$

The roots of an under-damped system are complex numbers. The Under-damped system does oscillate about the mean position, but unlike the undamped system, the amplitude of vibration dies down. Here the amplitude of vibration reduces slowly and diminished when mass stops oscillating about the equilibrium position.

For this condition, the roots S_1 and S_2 can be expressed as:

$$S_1 = (-\zeta + \mathrm{i}\sqrt{1-\zeta^2})\ \omega_n \tag{6}$$

$$S_1 = (-\zeta - \mathrm{i}\sqrt{1-\zeta^2})\ \omega_n \tag{7}$$

Where i is iota symbol, an imaginary solution was obtained as the $c^2 < 4mk$, and therefore, the value inside the square root becomes imaginary. Hence, imaginary roots were chosen.

The solution for an under-damped system is as obtained using Euler formula that converts the imaginary exponential equations to trigonometric functions:

$$x(t) = [C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)]e^{-\omega_n \zeta t} \tag{8}$$

$$C_1 = \frac{v_0 + \omega_n \zeta x_0}{\omega_d}$$

$$C_2 = x_0$$

$$\zeta = \frac{c}{2m\omega_n}$$

Here,

$$\omega_d = \sqrt{1-\zeta^2}\omega_n$$

ω_d is called the damped natural frequency of the system. The frequency of damped vibration w_d is always less than the undamped natural frequency ω_n. The decrease in the frequency of damped vibration with an increasing amount of damping can be seen from the equation. The under-damped case is crucial in the study of mechanical vibrations, as it is the only case that leads to oscillatory motion.

3.1.4 ζ = 1: Critically damped

For a critically damped system:

$$c^2 = 4mk\,(=c_c^2) \tag{9}$$

In this case, both roots are real and both equal $-\omega_n$. Critically damped systems are the only type of system that allows the fastest return to equilibrium position without any oscillation.

Because of the repeated roots, the solution would be:

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{10}$$

Where C_1 and C_2 are constants. The application of initial conditions, in this case, gives the values of constants as,

$$C_1 = x_0$$

$$C_2 = \dot{x}_0 + \omega_n x_0$$

3.1.5 ζ > 1: Overdamped

For overdamped system:

$$c^2 > 4mk$$

In this case, both roots are real and negative but not equal. The overdamped system moves toward the equilibrium position exponentially without oscillations.

The solution of the system can be expressed as:

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{11}$$

Where C_1 and C_2 are constants. The application of initial conditions, in this case, gives the values of constants as,

$$C_1 = \frac{x_0 \omega_n\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0 \omega_n\left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}$$

3.1.6 Comparison of Viscous damping cases:

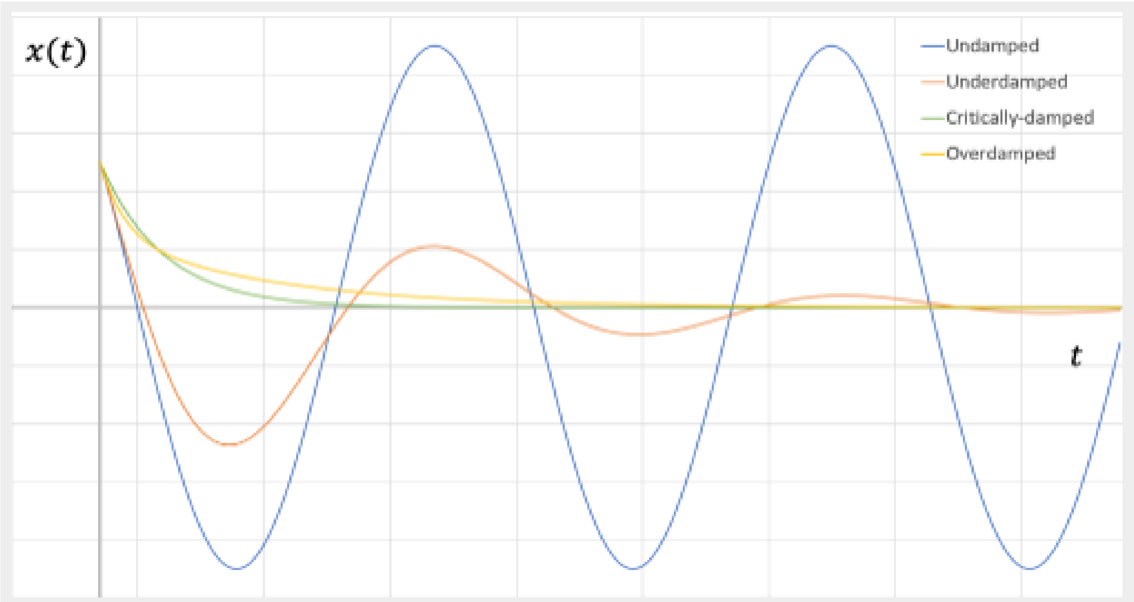

Figure 3.2 Comparison of Response of System

In the above figure, we can see that the critically damped response is the fastest, and over-damped response is the slowest response in the system returning to equilibrium. We can see that the underdamped system amplitude is quite attenuated compared to the undamped case.

3.2 MASS SPRING DRY COULOMB DAMPER SYSTEM

Sliding friction is the primary cause of energy dissipation in the Coulomb damping. The loss of energy occurs due to the comparative movement of the surfaces moving against one another. Coulomb damping was named after Charles-Augustin de Coulomb. There are two types of Coulomb damping: wet coulomb damping and dry coulomb damping. In this experiment, we will deal only with dry coulomb damping.

3.2.1 Modes of Coulomb damping:

Because of convenience and mechanical simplicity, coulomb dampers are used quite frequently in mechanical systems. In vibrating systems, whenever two surfaces slide against each other, coulomb damping appears internally. According to the Coulomb law of dry friction, "force required to produce sliding when two surfaces are in contact is directly proportional to the normal force acting in the plane of contact." Mathematically, the friction force is given by

$$\mathrm{F} = \mu\mathrm{N} = \mu\mathrm{W} = \mu\mathrm{mg}$$

Where W is the weight of the body, whereas μ is the frictional coefficient, for example, for the lubricated metal coefficient of friction is 0.1, for the nonlubricated metal coefficient of friction is 0.3.

Static damping occurs when the system does not undergo relative motion, or they are stationery.

$$F_s = \mu_s \cdot \mathrm{N} \tag{12}$$

Kinetic friction occurs when two bodies slide relative to each other.

$$F_k = \mu_k \cdot \mathrm{N} \tag{13}$$

3.2.2 Coulomb damping in the mass-spring system:

Two types of vibrations in the mass-spring system are discussed below:

3.2.2.1 *Free vibration in mass-spring system with Coulomb damping*

Consider a system as shown in figure below. The system is undergoing coulomb damping. We will discuss two cases as the friction force varies with velocity direction.

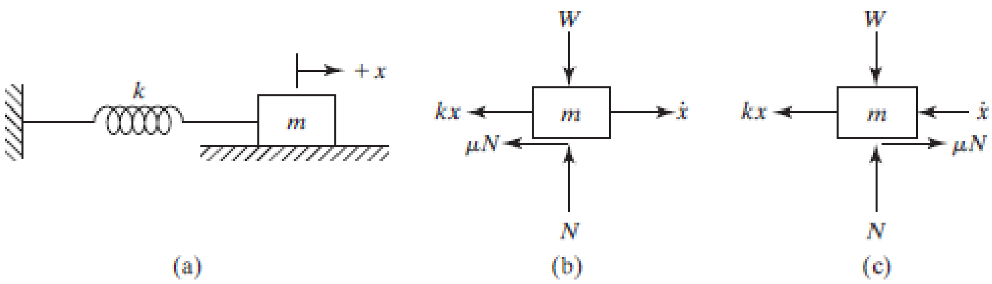

Figure 3.3 System in Coulomb Damping

Since the friction force varies with the direction of velocity, as shown in the below figure.

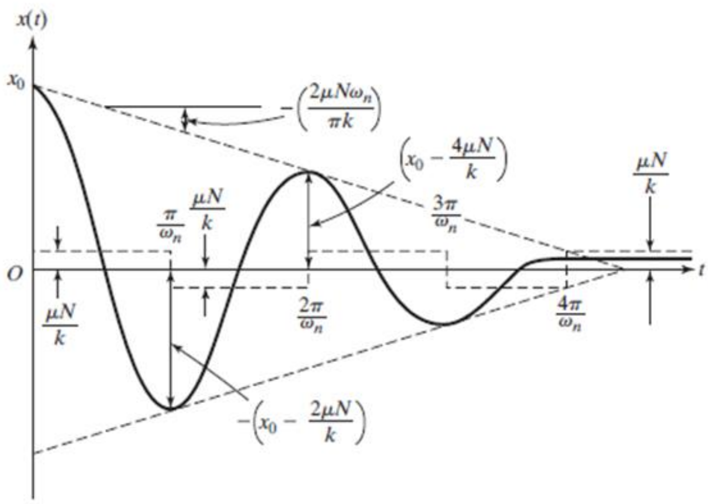

Figure 3.4 Motion of mass with Coulomb Damping

Case 1:

The first case is related to the positive value of displacement and its time derivative. When both values are positive, then the following differential equation is obtained.

$$m\frac{d^2x}{dt^2} = -kx - \mu N \mathrm{sign}(x)$$

It is a second-order non-homogenous differential equation. The solution to this equation is given as:

$$x(t) = A1 \cdot \cos(\omega_n)\,\mathrm{t} + A2 \cdot \cos(\omega_n)\,\mathrm{t} - \mu\mathrm{N/k} \qquad (14)$$

Where A_1 *and* A_2 are arbitrary constant.

$$\omega_n = \sqrt{\frac{\mathrm{k}}{\mathrm{m}}}$$

The amplitude of wave becomes zero when mass crosses the mean position. A_1 and A_2 are constants, and their value depends on the initial condition of the half-cycle.

Case 2:

The first case is related to the positive value of displacement and its time derivative is negative.

$$m \cdot \frac{d^2x}{dt^2} = -kx + \mu\mathrm{Nsign(x)}$$

It is a second-order non-homogenous differential equation. The solution of this equation is given as

$$x(t) = A1 \cdot \cos(\omega_n)\,\mathrm{t} + A2 \cdot \cos(\omega_n)\,\mathrm{t} + \mu\mathrm{N/k} \qquad (15)$$

Where the value of arbitrary coefficients was determined using the initial conditions.

$$A1 = xo - 3\mu \cdot \frac{\mathrm{N}}{\mathrm{k}}$$

$$A2 = 0$$

The term $\mu\mathrm{N/k}$ is a constant which shows the virtual displacement of spring under the action of force μN, if it is applied as a static force.

3.2.2.2 *The solution of Equation of Motions*

The system starts with zero velocity and displacement X_0 at t = 0. The motion starts from right to left as shown in figure (2.3). The value of A_1, A_2, A_3, A_4 can be calculated numerically or analytically. For a free vibration of the coulomb damped system, their values are calculated to be as follows.

$$A3 = xo - \mu \cdot \frac{\mathrm{N}}{\mathrm{k}}$$

$$A4 = 0$$

So, the solution for the half-cycle, i.e., $0 \leq t \leq \frac{\pi}{\omega_n}$ is

$$x(t) = (xo - \mu \cdot \frac{N}{k}) \cdot \cos(\omega_n)\, t + \mu N/k \tag{16}$$

For the second half cycle from $\frac{\pi}{\omega_n} \leq t \leq \frac{2\pi}{\omega_n}$, The values of constants are as follows [7].

$$-A1 = -xo + 3\mu \cdot \frac{N}{k}$$

$$A2 = 0$$

The steady-state solution is given by the following equation.

$$x(t) = \left(xo - 3\mu \cdot \frac{N}{k}\right) \cdot \cos(\omega_n)\, t - \mu N/k \tag{17}$$

For the number of half-cycles, after which the motion will cease can be calculated through the following equation.

$$r \geq \{\frac{x0 - \mu \frac{N}{k}}{2\mu \frac{N}{k}}\}$$

The response of the system undergoing free vibration under coulomb damping is shown in figure (3.4).

3.2.3 Characteristics of Coulomb damping:

Characteristics of Coulomb damping under free vibration are as follows.

- Equation of motion is nonlinear in the Coulomb damping case.
- With the change in Coulomb damping, the natural frequency of the system remains the same.
- Motion is periodic for coulomb damping.
- The system comes to rest with Coulomb damping after some time.
- Reduction in amplitude is linear with Coulomb damping.
- The reduction in amplitude is by 4uN/k after each successive cycle. So, the amplitude at the end of each consecutive cycle are related as:

$$X_m = X_{m-1} - \frac{4\mu N}{k}$$

3.3 SCILAB ODE SOLVER

Scilab ordinary differential equation (ODE) solver can be used to solve second-order or higher-order polynomial by transforming them into linear first-order differential equations via substitution. ODE Solver can be called using a built-in function ODE, which is given as below.

$$y = ode\ (y_0, t_0, t, f)$$

In this equation, y_o represent the initial conditions that are set for the solution of the ordinary differential equation. t_0 represents the time for which the initial condition was recorded. "t" represents the time period during which the function "f" will be differentiated using ordinary differential equations.

3.3.1 Viscous Damped ODE Solver Code

Viscous Damping solver code is attached in the appendix program 2. The program is used to record the response of the viscous underdamped system. In this program, the vibration response of the free damped system was carried out. For this purpose, the average mass of the system was selected 1.2 kilograms, and it is given by m=1.2. The stiffness constant of the spring was selected as 12.500 N/m, and the damping coefficient was selected as 1.6 Ns m^{-1}.

Moreover, the displacement and velocity at time t=0 were set as an initial condition. Displacement x at t=0 was 10 mm, and the initial velocity was chosen 0 at t=0. In this program, x (1) represents the displacement, while dx (1) and x (2) represent the velocity and displacement. The linear form of the ode equation is given by dx (2), which is the acceleration of the spring-mass system. Equations dx (1) and dx (2) will be used to solve the ODE function.

3.3.2 Coulomb Damping ODE Solver Code

This code is used to record the response of the Coulomb damped system. In this program, a dry damping system was used, and the arbitrary spring-mass system sliding on the flat surface was chosen for the study. The mass of 1.2 kilograms was attached to the spring of the stiffness 12.5N.m^{-1}. The surface static friction coefficient was designated by μ_s and it was selected as 0.5. It was assumed that the system is given an initial displacement of 10mm, and the initial velocity of the system was 0. In the program, the Jacobian function was used in the linear ODE. Coulomb Damping solver code is attached in the appendix under the heading program-1 attached x (1) represents the displacement, while dx (1) and x (2) represent the velocity. The linear form of the ode equation is given by dx (2), which is the acceleration of the spring-mass system. Equations dx (1) and dx (2) will be used to solve the ode function.

3.4 SCILAB FAST FOURIER TRANSFORM

In Scilab, Fast Fourier Transform is evaluated using the program-3 given in the appendix. The Program 'z' represents the vibration response of the system, which is in the form of sine waves. "fft" is used to calculate the fast Fourier transform of response 'z.' The result of the Fast Fourier transform is stored in a vector space "y_fft." Moreover, the frequency of the system is computed using the time period of the signal.

3.5 Q VALUE OF OSCILLATOR

It is a dimensionless number which is employed to relate the magnitude of damping of the oscillator. It is the ratio of the peak energy stored in a system during the cycle to the amount of energy lost per radian during the cycle. In the frequency domain, the quality factor is the ratio of the resonant frequency to the bandwidth of the signal. A higher value of Q indicates that system vibration will diminish quickly due to the significant damping provided to the system. It also shows that energy loss is maximum in such a system. On the other hand, a low value of Q indicates that the system will vibrate for a long time before coming to rest. It is because the energy loss is minimal due to the less damping available within the system.

3.5.1 Physical Significance of the Quality Factor

Physically, it is the factor of the ratio of energy stored in a cycle to the energy damped in a cycle. It is the rate of recurrence at which the system oscillates to the rate at which energy is lost by the system. It is obtained by employing the fast Fourier transform of the signal, which is then plotted against the amplitude and the frequency. In the frequency domain, the Quality of the oscillations is given by the formula as shown below.

$$Q = \frac{f_r}{\Delta f}$$

In this equation, numerator represents the peak or resonating frequency, while denominator indicates the frequency difference at the point where less than half of the signal power is attenuated [4]. It occurs typically at the difference of 3dB from the resonant frequency. Here 3dB is equivalent to the 0.7071 of the resonant value of the signal. [5]

3.5.2 Quality factor Vs. damping

3.5.2.1 *Overdamped System:*

In an over-damped system, since the damping coefficient is higher than the critical damping coefficient, therefore the vibration dies out immediately, and all the energy is lost quickly [10]. These systems show exponential decay, and the system returns to its stable form asymptotically. Moreover, since energy loss is high, so the bandwidth frequency of such systems is high; therefore, these systems have the Quality factor lesser than 0.5. [10]

3.5.2.2 *Underdamped System*

In the under-damped system, the damping coefficient of the damper is less than the critical damping value.

$$c < c_c$$

Therefore, the vibrations do not die out immediately, and the energy is lost sequentially in each cycle [6]. Such a system has the Quality factor greater than 0.5, and as the quality factor is increased, the damping reduces, and the high-quality dome shape is produced.

3.5.2.3 *Critically Damped System*

In these systems, the damping coefficient of the damper is equal to the critical damping coefficient of the system [9].

$$c = c_c$$

In these systems, the response is different from the critically and overdamped system. In these systems, the response does not overshoot, and output immediately reaches the stable value as in the underdamped system [10]. The Quality factor of such a system is 0.5.

3.6 MATHEMATICAL DERIVATION FOR COMBINE DAMPING

In the case of combine, damping, let us consider a free damped system. Where sliding friction coefficient $\mu \neq f(v)$, velocity independent. It is responsible for the creation of the Coulomb damping force F_D. Here, F_D is the damping force created due to the sliding friction [7]. This force varies inversely with the velocity.

The general equation of the motion of the system described above is given below:

$$\sum F_i = 0 = mx'' + Cx' + kx \pm F_D = 0 \tag{18}$$

As the block moves forward (left to right), the displacement and acceleration of the block are positive, while velocity decrease. Therefore, this force is directed toward the center, and we get the final form of the equation, as shown below.

$$\sum F_i = 0 = mx'' + Cx' + kx + F_D = 0 \tag{19}$$

In case when the block is moving toward the center (right to the left), the displacement of the block decrease and becomes negative. On the other hand, the velocity of the block increases, and the Coulomb damping force changes and becomes negative. This is given by the equation below.

$$\sum F_i = 0 = mx'' + Cx' + kx - F_D = 0 \tag{20}$$

The damping force is $F_D = c_0 \frac{x'}{|x'|}$ with $c_0 = \mu F_N = .\mu N$ The ratio of $\frac{x'}{|x'|}$ is velocity per absolute velocity. It implies that $\frac{x'}{|x'|} = 1$ for forwarding motion, $\frac{x'}{|x'|} = -1$ for backward motion and $x' = 0$ at rest, $\mu \to \mu_0$. Moreover, since sign($\dot{x}$) is a sign function whose values is -1 for $\dot{x}$<0 and its value is +1 for $\dot{x}$0 [2]. This property of the sign(x) function can be reflected using the relation below.

$$sign(x) = \frac{x'}{|x'|}$$

So, general $F_D = c_0 \frac{x'}{|x'|} = \mu\, F_N\, \frac{x'}{|x'|} = sign(x) \mu\, F_N$.

$$\sum F_i = 0 = mx'' + \mu\, F_N\, \frac{x'}{|x'|} + kx = 0$$

$$\sum F_i = 0 = mx'' + Cx' + \mu\, F_N sign(x') + kx = 0$$

$$x'' + \frac{C}{m}x' + \frac{\mu\, F_N}{m} sign(x') + \frac{k}{m}x = 0 \tag{21}$$

This equation was converted into a linear equation, and it was solved in the Scilab using an ordinary differential equation solver. For further solution by hand, this equation can be written in the non-homogeneous differential equation form, as shown below.

$$mx'' + Cx' + kx = \pm F_D$$

Now for such an equation, we will have two solutions homogeneous and solution as given below.

$$x(t) = x_h(t) + x_p(t)$$

The solution of the equation is given below,

$$x_p(t) = \pm \frac{F_D}{k}$$

In this equation $\frac{F_D}{k}$ is the constant representing the virtual displacement of the spring under the action of $\mu\, F_N$ if it were applied as a static force[2].

For a Homogeneous solution, the equation will be as shown below.

$$x'' + \frac{C}{m}x' + \frac{k}{m}x = 0$$

The general solution of the second-order homogeneous differential equation could be assumed of the form,

$$x(t) = ce^{st}$$

$$\dot{x}(t) = cse^{st}$$

$$\ddot{x}(t) = cs^2e^{st}$$

$$cs^2e^{st} + \left(\frac{c}{m}\right)cse^{st} + \left(\frac{k}{m}\right)ce^{st} = 0$$

Roots of the equation come out to be

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

Therefore, the homogeneous solution of the equation is shown below,

$$x_h(t) = C_1 \times e^{\left[\frac{-C+\sqrt{C^2-4mk}}{2m}\right]t} + C_2 \times e^{\left[\frac{-C-\sqrt{C^2-4mk}}{2m}\right]t}$$

The displacement of the mass is given by,

$$x(t) = \mathrm{C}_1 \times e^{\left[\frac{-C+\sqrt{C^2-4mk}}{2m}\right]t} + \mathrm{C}_2 \times e^{\left[\frac{-C-\sqrt{C^2-4mk}}{2m}\right]t} \pm \frac{F_D}{k} \tag{22}$$

Here C_1 and C_2 are the constant dependents on the initial conditions and the parameters such as Stiffens of the Spring, dampers damping coefficient, and critical damping coefficient of the system. While F_D/k is positive during the first half of the cycle when the block is moving toward the right, the equation can be written as shown below:

$$x(t) = e^{-\eta w_n t}\left(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)\right) + \frac{F_D}{k}$$

Now applying initial conditions, i.e., at t=0, the amplitude is x_0 velocity is v_0 and solving this we get

$$x(t) = e^{-\eta \omega_n t}\left(\left(x_0 - \frac{F_D}{k}\right)\cos(\omega_d t)\right) + \frac{F_D}{k}$$

Its value becomes negative when the block moves toward the left (during the negative part of the wave). During this phase, the equation is given below

$$x(t) = e^{-\eta \omega_n t}\left(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)\right) - \frac{F_D}{k}$$

At the end of the first half-wave, we obtain $t = \frac{T}{2} = \frac{\pi}{\omega_d}$. This leaves

$$x(t) = x\left(\frac{T}{2}\right) = \left(x_0 - \frac{F_D}{k}\right)\cos\left(\omega_d \frac{\pi}{\omega_d}\right) + \frac{F_D}{k} = -\left(x_0 - 2\,\frac{F_D}{k}\right)$$

During the second half-wave, the displacement of the system is given by the equation below.

$$x(t) = e^{-\eta \omega_n t}\left(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t)\right) - \frac{F_D}{k}$$

Finally, employing the initial condition, we get the final form of the equation for the second half, as shown below. These equations will be used in the theoretical calculation, apart from the numerical analysis.

$$x(t) = \left(x_0 - 3\,\frac{F_D}{k}\right)\cos(\omega_d t) - \frac{F_R}{k}$$

4 RESULT

4.1 MASS SPRING VISCOUS DAMPED SYSTEM

During this analysis, the mass and the stiffness of the system was kept constant, and the damping coefficient was varied. The natural angular frequency of the system was constant equal to the 37.699 rad/s, and the ratio of the damping coefficient to the mass "C/m" was varied from 31.416 to 6.283. The value was varied to find out its effect on the system. The initial displacement of the system was given as x_0=10mm, and the initial velocity was given as v_0=0.

Table 4-1 Parameters for Viscous Damped System

Mass (Kg)	**Spring constant (N/m)**	**Damping Coefficient (N s/m)**
20	28425	1.56-0.314

4.1.1 Displacement of the Viscous Damped System

The figure attached below shows the vibration response of the viscous damped system. During this response, the peak displacement is 10mm, and the vibration-damping show

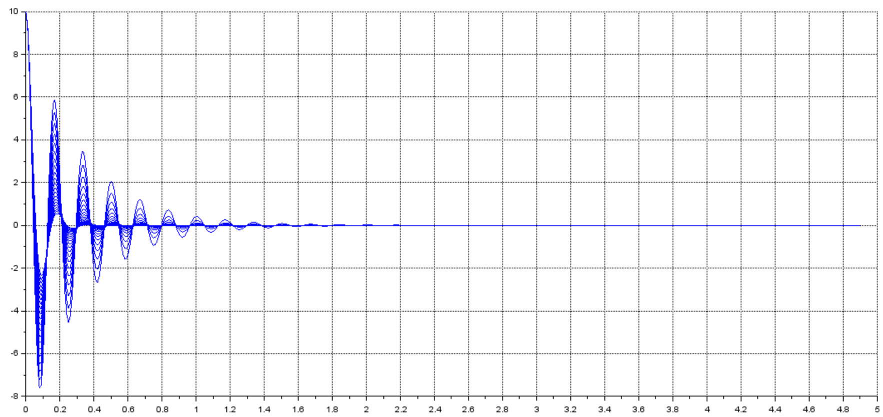

Figure 4.1 Response of Viscous Damped System

that the system is underdamped. The system vibrations are reducing slowly and gradually, and the system becomes stable after 2.5 seconds.

4.1.2 FFT Curve of the Viscous Damped System

The figure below shows the curve generated by taking the fast Fourier transform of the vibration response of the system. From the curve, there is a continuous increase in the bandwidth of the signal. Moreover, a single tone signal is formed during each cycle, and the resonance frequency is around 5 Hz. This bandwidth is increased because during the iteration, the damping coefficient is continuously decreased due to which the energy loss during each cycle decreases and the bandwidth decreases.

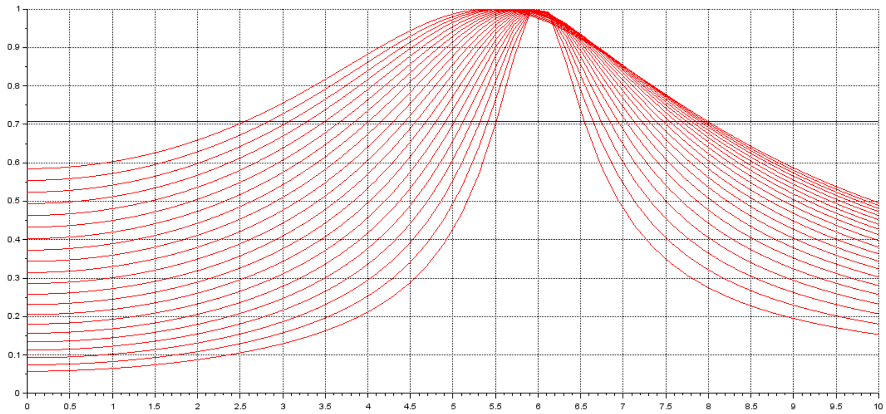

Figure 4.2 FFT of Viscous Damped System

4.1.3 Q vs. Damping Ratio

The figure below shows the variation of the Quality factor with the amplitude decay or damping ratio of the system. From the line graph, the Quality factor varies inversely with the amplitude decay factor. The graph shows that Q has the maximum value of the 5.712 against the damping ratio of 0.087524. Moreover, as the damping ratio increase, the Quality decreases, and the Q reaches its minimum value of 1.010 against the maximum damping ratio value 0.494537. Therefore, the range of damping ratio was 0.0875 to 0.4945.

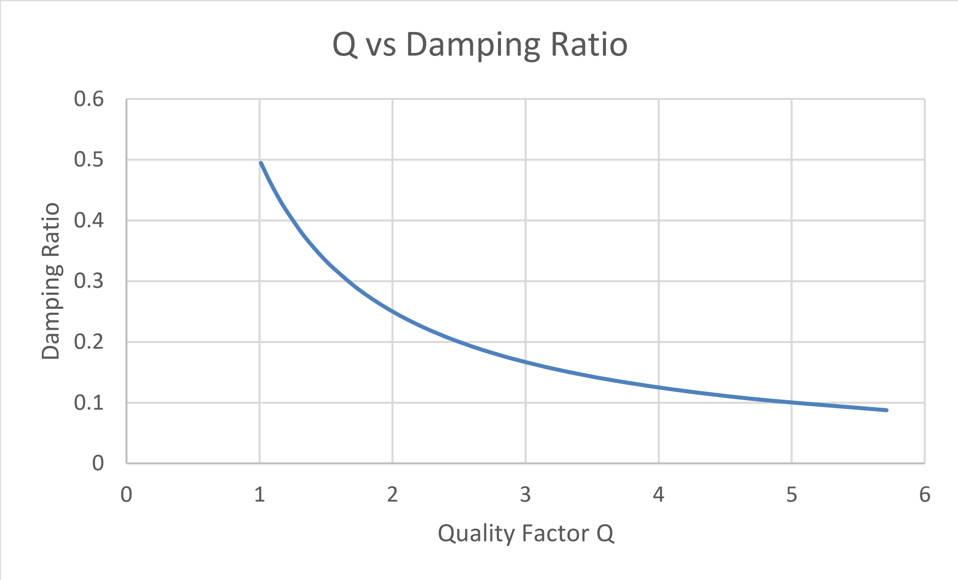

Figure 4.3 Q vs. Damping Ratio for Viscous Damped System

4.1.4 Q vs. Bandwidth

The figure below shows the variation of the Quality factor against the bandwidth frequency. The line graph illustrates the inverse relation of the Q and bandwidth. The Q has the maximum value of the 5.71 against bandwidth frequency of 1.036. An increase in the bandwidth frequency leads to the drop in Q value, and finally, the Q values become equal to 1.011 for the bandwidth frequency of 5.45.

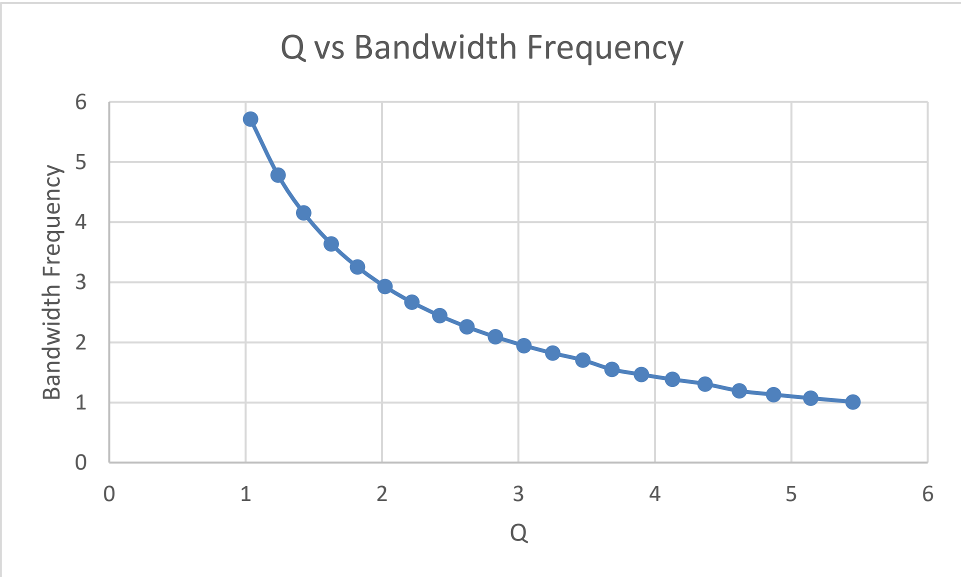

Figure 4.4 Q vs. Bandwidth Frequency for Viscous Damped System

4.2 SPRING MASS COLUMB DAMPED SYSTEM

During this analysis, the mass and the stiffness of the system was kept constant along with the damping coefficient. The natural angular frequency was constant equal to the 31.415 rad/s. Its value is selected smaller because energy loss in coulomb damping is smaller than the viscous damped system. So, the natural system frequency was decreased to improve the damping ratio of the Coulomb damped system. The ratio of the damping coefficient to the mass "C/m" was varied from 25.13 to 6.283. The initial displacement of the system was given as x_0=10mm, and the initial velocity was given as v_0=0. As per the requirement, the response was calculated for the time of 7.8 seconds.

Table 4-2 Parameter for Coulomb Damped System

Mass (Kg)	Spring constant (N/m)
20	19738

4.2.1 Displacement of the Coulomb Damped System

The figure attached below shows the vibration response of the Coulomb damped system. During this response, the peak displacement is 10mm, and the vibration-damping show that the drop-in amplitude is linear. The system vibrations are reducing linearly and gradually reaching the minimum displacement of the 9.5mm in 7.8sec. These results were validated by the previous research carried out for the Coulomb damping.

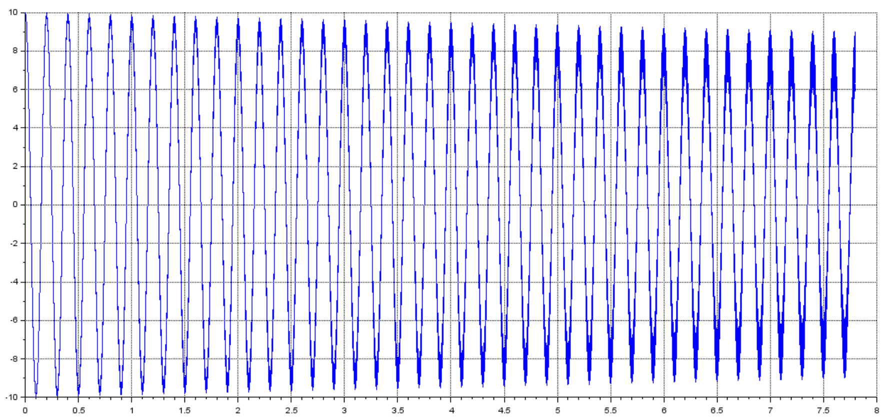

Figure 4.5 Response of Coulomb Damped System

4.2.2 FFT Curve of the Coulomb Damped System

The figure below shows the curve generated by taking the fast Fourier transform of the vibration response of the Coulomb damped system. From the curve, there unlike the viscous damped system, there is an only minor increase in bandwidth of the signal. Moreover, a single tone signal is formed during each cycle, and the resonance frequency is around 5 Hz. This bandwidth has remained approximately constant even though the friction coefficient is continuously decreased due to which the energy loss during each cycle decreases.

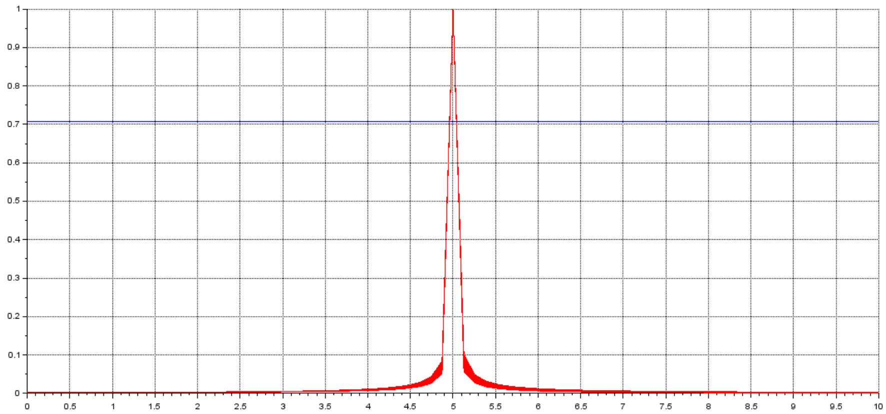

Figure 4.6 FFT of Coulomb Damped System

4.2.3 Q vs. Damping Ratio

Although the damping ratio is a term related explicitly to viscous damping, however, an indirect relation can be derived to calculate it for the Coulomb damping case too. Since derived before, the damping ratio is the ratio of damping of the system to the critical value of the damping of the system. Since it is a ratio, it has no units. This ratio determines three cases of motion. Underdamping if this ratio is less than 1, overdamping if this ratio is greater than one and critical damping if this ratio is 1. The purpose of showing the damping ratio here, although we cannot calculate it directly, is to understand the difference between vicious and coulomb damping magnitudes.

The damping ratio is related to quality factors by the relationship in the following [2].

$$Q = \frac{1}{2\zeta}$$

Where $Q = Quality\ Factor,\ \zeta = Damping\ Ratio$

The figure below shows the variation of the Quality factor with the damping ratio of the system. From the line graph, as the Quality factor increases, there is a slight decrease in the amplitude or decay factor. The graph shows that as the damping ratio increase, the Quality decreases, and the Q reaches its minimum value of the 60.386 against the maximum damping ratio value of 0.00828. The graph also indicates the small variation in the values of the quality factor with the damping ratio. However, the declining trend of the Quality factor with the increase in amplitude decay is a dominant factor.

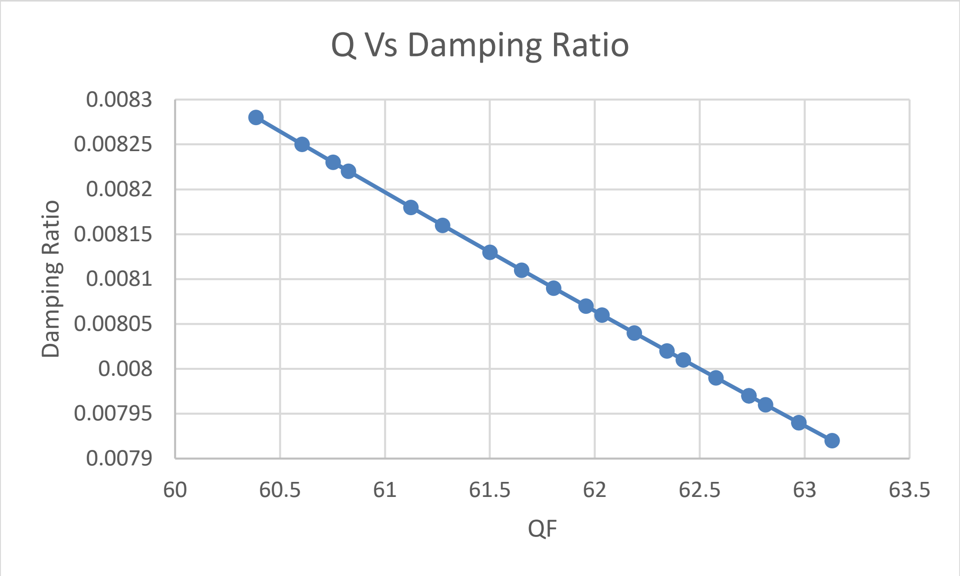

Figure 4.7 Q vs. Damping Ratio of Coulomb Damped System

4.2.4 Q vs. Bandwidth

The figure below shows the variation of the Quality factor against the bandwidth frequency. The line graph illustrates the inverse relation of the Q and bandwidth. The Q has the maximum value of the 63.13 against bandwidth frequency of 0.079. An increase in the bandwidth frequency leads to the drop in Q value, and finally, the Q values become equal to 60.328 for the bandwidth frequency difference of 0.0828.

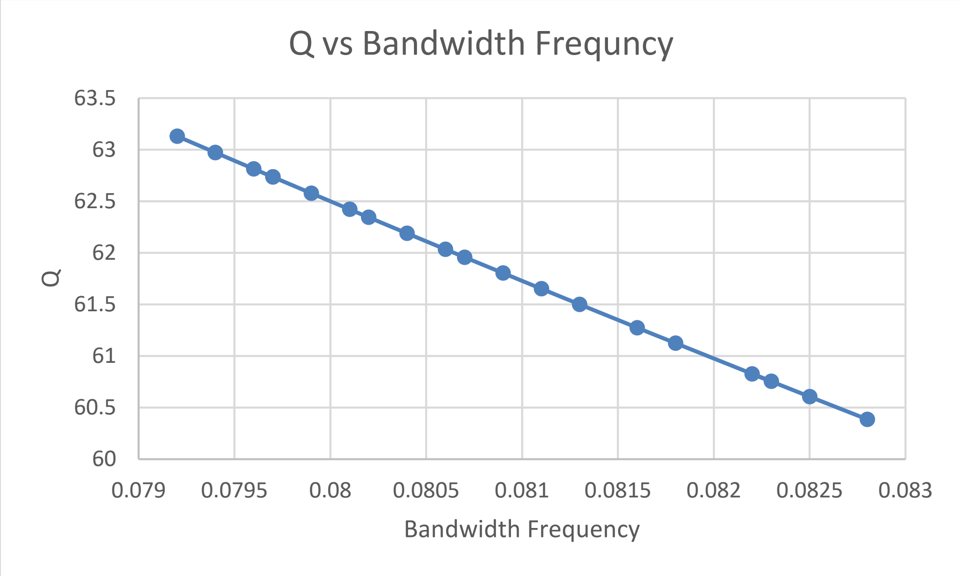

Figure 4.8 QF vs. Bandwidth frequency of Coulomb Damped System

Bandwidth is taken as the difference of curves at 0.7 in the fast Fourier transform plot in figure 3.6. Since this bandwidth is very small, the difference is in 10^{-2}.

This linear graph is obtained because the vibrations are not entirely damped. This relation linear relation is because since surface static friction coefficient is constant, and the damping rate is minimal. The figure below shows the calculation and the results for the Coulomb damped system.

TWICE_KSI_TIMES_OMEGA	OMEGA	NUMERIC_F_DAMPED	f_up	f_down	Bandwidth	Q_numeric	Damping Ratio
25.13274123	31.41592654	5	5.0418	4.959	0.0828	60.386473	0.00828
24.19026343	31.41592654	5	5.0417	4.9592	0.0825	60.606061	0.00825
23.24778564	31.41592654	5	5.0412	4.959	0.0822	60.827251	0.00822
22.30530784	31.41592654	5	5.0413	4.959	0.0823	60.753341	0.00823
21.36283004	31.41592654	5	5.0412	4.9594	0.0818	61.124694	0.00818
20.42035225	31.41592654	5	5.0411	4.9595	0.0816	61.27451	0.00816
19.47787445	31.41592654	5	5.0409	4.9596	0.0813	61.500615	0.00813
18.53539666	31.41592654	5	5.0408	4.9597	0.0811	61.652281	0.00811
17.59291886	31.41592654	5	5.0407	4.9598	0.0809	61.804697	0.00809
16.65044106	31.41592654	5	5.0406	4.9599	0.0807	61.957869	0.00807
15.70796327	31.41592654	5	5.0405	4.9599	0.0806	62.034739	0.00806
14.76548547	31.41592654	5	5.0404	4.96	0.0804	62.189055	0.00804
13.82300768	31.41592654	5	5.0403	4.9601	0.0802	62.34414	0.00802
12.88052988	31.41592654	5	5.0402	4.9601	0.0801	62.421973	0.00801
11.93805208	31.41592654	5	5.0401	4.9602	0.0799	62.578223	0.00799
10.99557429	31.41592654	5	5.04	4.9603	0.0797	62.735257	0.00797
10.05309649	31.41592654	5	5.04	4.9603	0.0797	62.735257	0.00797
9.110618695	31.41592654	5	5.0399	4.9603	0.0796	62.81407	0.00796
8.168140899	31.41592654	5	5.0398	4.9604	0.0794	62.972292	0.00794
7.225663103	31.41592654	5	5.0398	4.9604	0.0794	62.972292	0.00794
6.283185307	31.41592654	5	5.0397	4.9605	0.0792	63.131313	0.00792

Figure 4.9 Calculations for Coulomb Damped System

4.3 SPRING MASS COMBINED DAMPED SYSTEM

During this analysis, the mass and the stiffness of the system was kept constant. While the damping coefficient and coefficient of static friction were varied. The natural angular frequency of the system was kept constant equal to the 37.699 rad/s, and the ratio of the damping coefficient to the mass "C/m" was varied from 31.415 to 6.283 randomly. The initial displacement of the system was given as x_0=10mm, and the initial velocity was given as v_0=0. As per the requirement, the response was calculated for the time of 10 seconds, and the response was obtained identical to the Coulomb damped system response [1].

4.3.1 Displacement of the Combined Coulomb-Viscous Damped System

The figure attached below shows the vibration response of the combined damped system. During this response, the peak displacement is 10mm, and the vibration-damping show the cumulative effect of the viscous and coulomb damping. The vibration in this system was damped. The system vibrations are reducing quickly, and the system reaches a stable state within less than 1 second. Moreover, there was distortion in the vibration waves due to the combined effect of the viscous and coulomb damped system.

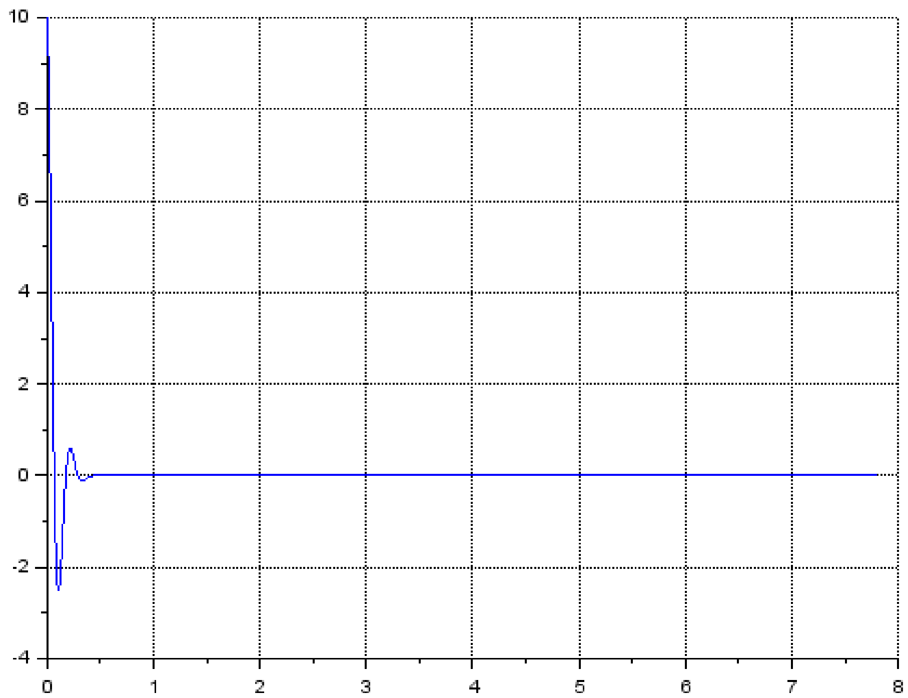

Figure 4.10 Combine Damped System Response

4.3.2 FFT Curve of the Combine Damped System

The figure below shows the curve generated by taking the fast Fourier transform of the vibration response of the combined viscous and coulomb damped system. From the curve, there unlike the viscous damped system, there is an only minor increase in bandwidth of the signal. Moreover, a single tone signal is formed during each cycle, and the resonance frequency is 4.4872 Hz.

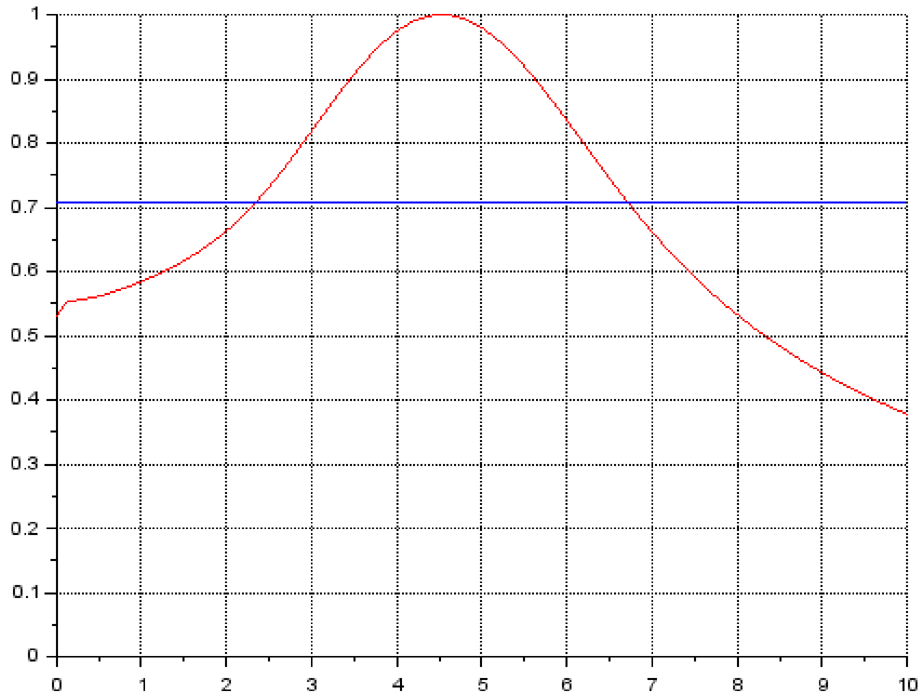

Figure 4.11 FFT Curve of Combine Damped System

4.3.3 Q vs. Amplitude Decay

The figure below shows the variation of the Quality factor with the damping ratio of the system. The line graph indicated the impact of vibration damping. Moreover, it also verifies the relation that the Quality factor varies inversely with the amplitude decay factor or damping ratio. The graph shows that initially, the Q value is smaller, and the damping ratio is larger, and the variation of the quality factor with the damping ratio is significant. However, as the quality factor increase, the curve becomes flat and drops to the minimum value of 0.0166 for the quality factor of 30.196. The amplitude decays exponentially with time i.e. $e^{-\eta\omega_n t}$. This term includes the combined damping effects of viscous and coulomb damping. The damping ratios of viscous and coulomb damping are shown in previous sections in Figures 3.3 and 3.7. The viscous damping has a higher amount of damping ratio than coulomb damping. Therefore, viscous damping dominates the process primarily.

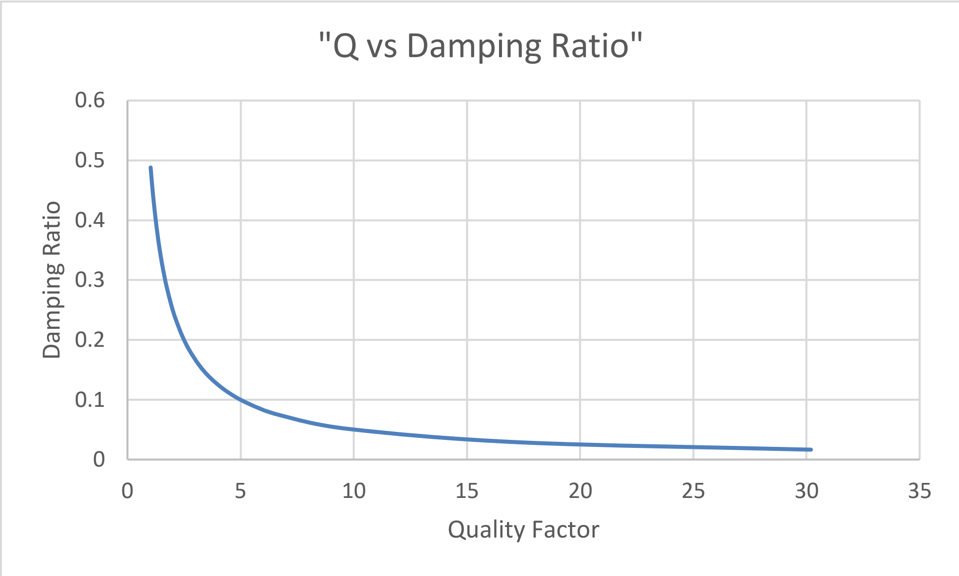

Figure 4.12 Q vs. Damping Ratio of Combine Damped System

4.3.4 Q vs. Bandwidth

The figure below shows the variation of the Quality factor against the bandwidth frequency. The line graph illustrates the inverse relation of the Q and bandwidth.

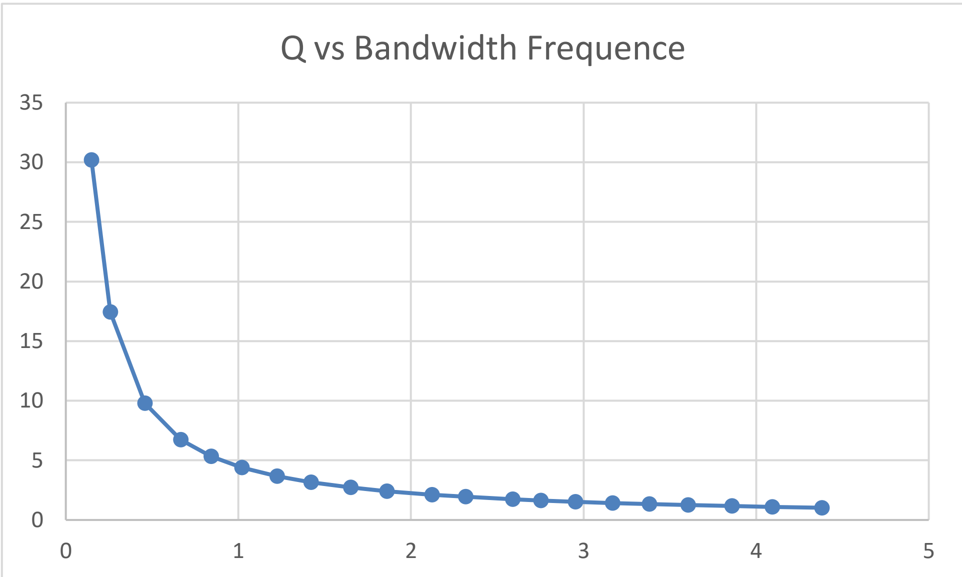

Figure 4.13 Q vs. Bandwidth Frequency of Combine Damped System

This bandwidth frequency is the difference between the rising end and lowering end frequency at the value 0.707 of the peak frequency. The Q has the maximum value of the 30.19 against bandwidth frequency of 0.1486. An increase in the bandwidth frequency leads to a drop in Q value, and finally, the Q values become equal to 1.0243 for the band-width frequency difference of 4.3805.

5 DISCUSSION AND CONCLUSION

The vibration response was obtained for a damped system by employing analytical techniques. Besides, to this, the response produced from the analysis was processed to quantify the natural frequency of the system under observation. The fast Fourier transform technique was employed for this purpose [11]. The Frequency response of the system resembles the available data set of frequency responses. Moreover, a single peak was obtained, clearly indicating the natural frequency of the system [12]. In addition to this following significant points were identified.

- The frequency response of the viscous damped system was identical to the frequency response of the electrical analog RLC series circuit.
- The bandwidth in the viscous damping was higher than the bandwidth of the Coulomb damping. This result justifies the point discussed by [13] in that more energy is a loss in the viscous damped system as compared to the Coulomb damped system.
- The Quality factor for the Coulomb damped system varies from the quality factor of the viscous the damped system by the range 98.3% to 90.95%. This justifies the point discussed by [14] that less energy dissipation in the Coulomb damped system as compared to the viscous damped system.
- The response of the combined damped system shows the dominance of the viscous damping in damping the system vibration. A sine wave was achieved with the decay curve resembling closely to the overdamped system.
- The frequency response of the combined damped system is identical to the frequency response analysis result carried out by [15].

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7 APPENDIX

7.1 PROGRAM-1

```
function dx=f(t, x)
    m=1.2; //Kilograms
    C=1.600; //N s/m
    K=12.500; //Nm^-1
    dx(1)=x(2);
    dx(2)=-(C/m)*x(2)-(K/m)*x(1);
endfunction
clf()
t=0:0.01:10;
sol=ode([10;0],0,t,f);
z=sol(1,:);
xset("window",3)
xgrid(0)
plot2d(t,z,2)
xgrid()
```

7.2 PROGRAM-2

```
function dx=f1(t, x)
    m=1.200; //Kilograms
    K=12.5; //Nm^-1
    mu_s=0.5;  // Static Coefficient of Friction
    dx(1)=x(2);
    dx(2)=-(mu_s*9.81)*(sign(x(2)))-(K/m)*x(1);
```

```
endfunction

function J=Jacobian(t, y)

   J=A

Endfunction

clf()

t=0:0.01:5;

sol=ode("rk",[5;0],0,t,f1);

z=sol(1,:);
```

7.3 PROGRAM-3

```
//compute the fft

y_fft=fft(z,-1);

mag=fftshift(abs(y_fft));

frequency=0:1 ./max(t):(max(size(t))-1)/max(t);
```